Stabilization of Uncertain Negative-Imaginary Systems Using a Riccati Equation Approach

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Applications of flexible structure

Preliminary result

Aim of this work

Main results

Illustrative Example

Conclusion

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Systems contain flexible structure
Flexible structure

- The problem of vibration control for flexible structures arises in a variety of aerospace applications such as large space structures and flexible dynamics in air vehicles.
- These problems also arise in other areas of advanced technology such as nano-positioning, the control of atomic force microscopes and control of optical interferometric sensing systems.
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Control of flexible structures with collocated force actuators and velocity measurements, called negative-velocity feedback (Positive real systems).

Control of flexible structures with collocated force actuators and position sensors called positive-position feedback (Negative imaginary systems).

Positive-position feedback controller can be designed to increase the damping of the modes of a flexible structure.
Control of flexible structure

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Control of flexible structures

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- Positive-position feedback controller can be designed to increase the damping of the modes of a flexible structure.
Definition

A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1. $G(s)$ has no pole at the origin and in $Re[s] > 0$.
2. For all $\omega > 0$, such that $j\omega$ is not a pole of $G(s)$, and $j(G(j\omega) - G(j\omega)^*) \geq 0$.
3. If $j\omega_0$, is a pole of $G(j\omega)$, it is at most a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0} (s - j\omega_0)sG(s)$ is positive semidefinite Hermitian.
Generalized NI definition (Mabrok2012)

**Definition**

A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1. $G(s)$ has no pole in $\text{Re}[s] > 0$.
2. For all $\omega \geq 0$ such that $j\omega$ is not a pole of $G(s)$, $j(G(j\omega) - G(j\omega)^*) \geq 0$.
3. If $s = j\omega_0, \omega_0 > 0$ is a pole of $G(s)$ then it is a simple pole. Furthermore, if $s = j\omega_0, \omega_0 > 0$ is a pole of $G(s)$, the residual matrix $K = \lim_{s \to j\omega_0} (s - j\omega_0)jG(s)$ is positive semidefinite Hermitian. If $s = 0$ is a pole of $G(s)$, then $G_t = \lim_{s \to 0} s^t G(s) = 0$ for all $t \geq 3$ with $G_2$ is positive semidefinite Hermitian.
Negative imaginary Lemma

**Lemma**

Let \[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\] be a minimal realization of the transfer function matrix \(G(s) \in \mathbb{R}^{m \times m}\). Then, \(G(s)\) is NI if and only if there exist matrices \(P = P^T \geq 0, W \in \mathbb{R}^{m \times m}\), and \(L \in \mathbb{R}^{m \times n}\) such that the following LMI is satisfied:

\[
\begin{bmatrix}
PA + A^T P & PB - A^T C^T \\
B^T P - CA & -(CB + B^T C^T)
\end{bmatrix}
\begin{bmatrix}
-L^T L & -L^T W \\
-W^T L & -W^T W
\end{bmatrix}
\leq 0.
\] (1)
Definition

A square transfer function matrix $G(s)$ is SNI if the following conditions are satisfied:

1. $G(s)$ has no pole in $\text{Re}[s] \geq 0$.
2. For all $\omega > 0$, $j (G(j\omega) - G(j\omega)^*) > 0$. 

Strictly negative imaginary systems
**Stability result**

**Figure:** Negative-imaginary feedback control system. If the plant transfer function matrix $G_2(s)$ is strictly negative imaginary and the controller transfer function matrix $G_1(s)$ with no poles at the origin is negative imaginary, then the closed-loop system is internally stable if and only if the DC gain condition $\lambda_{\text{max}}(G_1(0)G_2(0)) < 1$ is satisfied.

**Theorem**

Consider an NI transfer function matrix $G_1(s)$ with no poles at the origin and an SNI transfer function matrix $G_2(s)$, (see Fig. 1), and suppose that $G_1(\infty)G_2(\infty) = 0$ and $G_2(\infty) \geq 0$. Then, the positive-feedback interconnection of $G_1(s)$ and $G_2(s)$ is internally stable if and only if $\lambda_{\text{max}}(G_1(0)G_2(0)) < 1$. 
Aim of this work

- Deriving the NI lemma in algebraic Riccati equation form.
- Providing a static controller synthesis method based on algebraic Riccati equation.
Main results

**Theorem**

Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a minimal realization with $CB + B^T C^T > 0$. Then $G(s)$ is NI if and only if $D = D^T$ and there exists a matrix $P \geq 0$ such that $P$ is a solution to the following algebraic Riccati equation

\[ PA_0 + A_0^T P + PBR^{-1}B^TP + Q = 0, \]  

(2)

where

\[
A_0 = A - BR^{-1}CA, \\
R = CB + B^T C^T, \quad \text{and} \\
Q = A^T C^T R^{-1} CA.
\]
Main results 2

Theorem

Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a minimal realization and $CB + B^T C^T > 0$. Then $G(s)$ is SNI if and only if

1. A has no $j\omega$-axis eigenvalues and $D = D^T$,
2. there exists a matrix $P > 0$ such that $P$ is a solution to the following algebraic Riccati equation

$$PA_0 + A_0^T P + PBR^{-1} B^T P + Q = 0,$$

(3)

where all the eigenvalues of the matrix $A_0 + BR^{-1} B^T P$ lie in the open left half of the complex plane or at the origin.
In order to present a synthesis result, consider the following state space representation for a linear uncertain system

\[
\begin{align*}
\dot{x} &= Ax + B_1 w + B_2 u, \\
z &= C_1 x, \\
w &= \Delta(s)z,
\end{align*}
\]

(4)

where, \(A \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^{n \times m}, B_2 \in \mathbb{R}^{n \times r}, C_1 \in \mathbb{R}^{m \times n},\) and \(\Delta(s)\) represents the uncertainty matrix.
Suppose that $K$ is a static controller such that $u = Kx$. Then the closed-loop interconnection of the system (4) with the static controller $K$ is given by:

$$
\begin{align*}
\dot{x} &= (A + B_2K)x + B_1w, \\
z &= C_1x, \\
w &= \Delta(s)z.
\end{align*}
$$

Our aim is to construct the controller $K$ such that the corresponding closed-loop system (5) satisfies the NI property.
Main results 3

Theorem

Consider an uncertain system model as in (4) with $C_1B_2$ invertible and $R = C_1B_1 + B_1^TC_1^T > 0$. Then there exists a controller $K$ such that the closed-loop system in (5) is NI if there exist matrices $T \geq 0$ and $S \geq 0$ such that

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0,$$

$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0$$

and $S - T < 0$. Here, $A_{22}$ is the anti-stable block of the matrix $A_f = U^T(A - B_2(C_1B_2)^{-1}C_1A)U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Furthermore, the controller gain matrix is

$$K = (C_1B_2)^{-1}(B_1^TP - C_1A - R(B_2^TC_1^T)^{-1}B_2^TP),$$

where $P = UP_fU^T$ and $P_f = \begin{bmatrix} 0 & 0 \\ 0 & -(S - T)^{-1} \end{bmatrix}$. 
Consider the following uncertain system of the general form (4) where,

\[
A = \begin{bmatrix}
-1 & 0 & -1 \\
1 & 1 & -1 \\
-5 & 1 & 1
\end{bmatrix};
B_1 = \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix};
B_2 = \begin{bmatrix}
0 \\
4 \\
2
\end{bmatrix};
C_1 = \begin{bmatrix}
0 & 2 & -3
\end{bmatrix}.
\]
This system satisfies Theorem 8. Applying Schur decomposition to the matrix \((A - B_2(C_1B_2)^{-1}C_1A)\) gives
\[
A_f = \begin{bmatrix}
-2.8557 & -3.5104 & -39.4820 \\
0 & 0 & 2.8020 \\
0 & 0 & 10.8557
\end{bmatrix}.
\]
The solution to Lyapunov equations gives \(T = 0.0156\) and \(S = 0.0120\) which implies that \(X = -0.0036\). It follows from Theorem 8 that \(P_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 275.5235 \end{bmatrix} \geq 0\). This implies that the controller gain matrix (8) is given by \(K = \begin{bmatrix} 140.2792 & -14.2713 & -55.5191 \end{bmatrix}\), According to Theorem 8, the closed-loop feedback system (5) from \(w\) to \(z\) is NI. To illustrate this we plot the imaginary part of the transfer function matrix of the closed-loop system from \(w\) to \(z\) in Fig. 3.
Applications of flexible structure

Preliminary result

Aim of this work

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Figure: Plot of the imaginary part of the closed-loop system.
The algebraic Riccati equation approach was used to derive a negative imaginary (NI) lemma and a strict negative imaginary (SNI) lemma.

The NI lemma was employed to solve a negative imaginary controller synthesis problem for an uncertain system.

A static controller was chosen to force the plant to be stable and satisfy the negative imaginary property under certain assumptions. This controller can be used to guarantee the robustness stability of the closed-loop system with strict negative imaginary uncertainty.
Thank you for your attention :)