Ensemble Control Synthesis via the SVD

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Motivation

- Ensemble control: Guidance of structurally identical systems, indexed by a varying parameter set, between desired configurations, using a common control
- Robust (to parameter variation) open loop control
- Methods: pseudospectral discretization, analytical solution
- Why direct spectral methods?
  - Fast: do not require optimization steps.
  - Powerful: control tasks requiring long inputs are tractable
  - Robust: iterative methods converge quickly if regular solutions exist
- Why *not* spectral methods?
  - Not general enough: formulation difficult for constrained problems
  - Numerical accuracy and stability are topics of current research.
Finite-dimensional time-varying linear ensemble system

Consider a parameterized family of dynamical systems indexed by $\beta \in K$ compact,

$$\dot{X}(t, \beta) = A(t, \beta)X(t, \beta) + B(t, \beta)u(t), \quad (\Sigma)$$

where $A(t, \beta) \in \mathbb{R}^{n \times n}$ and $B(t, \beta) \in \mathbb{R}^{n \times m}$ have elements that are real $L_\infty$ and $L_2$ functions, respectively, on compact set $D = [0, T] \times K$. We write $A \in L_\infty^{n \times n}(D)$ and $B \in L_2^{n \times m}(D)$. 
Let $\mathcal{H}_T = L^m_2[0, T]$ denote the set of $m$-tuples, whose elements are vector-valued square-integrable measurable functions defined on $0 \leq t \leq T$, with an inner product defined by

$$\langle g, h \rangle_T = \int_{0}^{T} g'(t)h(t) dt,$$

where $'$ denotes the transpose. Similarly, let $\mathcal{H}_K = L^n_2(K)$ be equipped with an inner product

$$\langle p, q \rangle_K = \int_{K} p'(\beta)q(\beta) d\mu(\beta),$$

where $\mu$ is the Lebesgue measure. With well-defined addition and scalar multiplication, $\mathcal{H}_T$ and $\mathcal{H}_K$ are separable Hilbert spaces, where $\| \cdot \|_T$ and $\| \cdot \|_K$ denote their respective induced norms.
Definition: (Ensemble Controllability)

We say that the family \((\Sigma)\) is ensemble controllable on the function space \(\mathcal{H}_K\) if for all \(\varepsilon > 0\), and all \(X_0, X_F \in \mathcal{H}_K\), there exists \(T > 0\) and an open loop piecewise-continuous control \(u \in \mathcal{H}_T\), such that starting from \(X(0, \beta) = X_0(\beta)\), the final state \(X(T, \beta) = X_T(\beta)\) satisfies

\[\|X_T - X_F\|_K < \varepsilon.\]

In other words, the system \((\Sigma)\) is ensemble controllable if it is possible to guide it from \(X_0\) to \(X_F\) in the space \(\mathcal{H}_K\), where the acceptable range of \(T \in (0, \infty)\) may depend on \(\varepsilon\), \(K\), and \(U\).
System Dynamics Operator

Given initial state $X(0, \beta) = X_0(\beta)$ of the system $(\Sigma)$, the variation of parameters formula yields

$$X(T, \beta) = \Phi(T, 0, \beta)X_0(\beta) + \int_0^T \Phi(T, \sigma, \beta)B(\sigma, \beta)u(\sigma)d\sigma,$$

where $\Phi(T, 0, \beta)$ is the transition matrix for the system $\dot{X}(t, \beta) = A(t, \beta)X(t, \beta)$. Setting $X(T, \beta) = X_F(\beta)$, pre-multiplying by $\Phi(0, T, \beta)$ and rearranging yields

$$(Lu)(\beta) = \int_0^T \Phi(0, \sigma, \beta)B(\sigma, \beta)u(\sigma)d\sigma = \xi(\beta),$$

where $\xi(\beta) = \Phi(0, T, \beta)X_F(\beta) - X_0(\beta)$. 
Definition: (Singular System)

Let \( Y \) and \( Z \) be Hilbert spaces and \( L : Y \to Z \) be a compact operator. If \( (\sigma_n^2, \nu_n) \) is an eigensystem of \( LL^* \) and \( (\sigma_n^2, \mu_n) \) is an eigensystem of \( L^*L \), namely, \( LL^*\nu_n = \sigma_n^2\nu_n, \nu_n \in Z \), and \( L^*L\mu_n = \sigma_n^2\mu_n, \mu_n \in Y \), where \( \sigma_n > 0 \ (n \geq 1) \), and the two systems are related by the equations \( L\mu_n = \sigma_n\nu_n \) and \( L^*\nu_n = \sigma_n\mu_n \), we say that \( (\sigma_n, \mu_n, \nu_n) \) is a singular system of \( L \).
Ensemble controllability conditions

Let \((\sigma_n, \mu_n, \nu_n)\) be a singular system of the compact operator \(L\). Necessary & sufficient conditions for controllability of \((\Sigma)\) are

\[(i)\quad \sum_{n=1}^{\infty} \left| \frac{\langle \xi, \nu_n \rangle K}{\sigma_n^2} \right|^2 \sigma_n^2 < \infty, \]

\[(ii)\quad \xi \in \overline{R(L)}, \]

where \(\overline{R(L)}\) is the closure of the range space of \(L\). The control law

\[ u = \sum_{n=1}^{\infty} \frac{\langle \xi, \nu_n \rangle K}{\sigma_n} \mu_n \]

satisfies \(\langle u, u \rangle_T \leq \langle u_0, u_0 \rangle_T \) for all \(u_0 \in \{v \mid Lv = \xi, v \neq u\}\).

Matrix Approximation

Let \( \{\beta_j\}, j = 0, 1, \ldots, P - 1 \) be finite, uniformly sampling \( K \). Let \( \{t_k\}, k = 0, 1, \ldots, N \) linearly interpolate \( [0, T] \) with \( t_k - t_{k-1} = \delta \).

\[
(Lg)(\beta) = \int_0^T \Phi(0, t, \beta) B(t, \beta) g(t) dt = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \Phi(0, t, \beta) B(t, \beta) g(t) dt
\]

\[
\approx \sum_{k=1}^N \delta \Phi(0, t_k, \beta) B(t_k, \beta) g(t_k).
\]

The action of the \( L : \mathcal{H}_T \to \mathcal{H}_K \) on \( g \in \mathcal{H}_T \) is approximated by action of \( W \in \mathbb{R}^{nP \times mN} \), with \( n \times m \) blocks \( W_{jk} = \delta \Phi(0, t_k, \beta_j) B(t_k, \beta_j) \), on a vector \( \hat{g} \in \mathbb{R}^{mN} \), with \( N \) blocks \( \hat{g}_k = g(t_k) \) of dimension \( m \times 1 \).

\[
(Lg)(\beta) \approx \delta \left[ \Phi(0, t_1, \beta) B(t_1, \beta) \cdots \Phi(0, t_N, \beta) B(t_N, \beta) \right] \hat{g}
\]
Singular Value Decomposition

- SVD is \( W = U \Sigma V' \), and \( \bar{u}_j \) and \( \bar{v}_j \) are columns of \( U \) and \( V \), respectively, corresponding to the singular value \( s_j \).
- \( WW'\bar{u}_j = s_j^2 \bar{u}_j \) and \( W'W\bar{v}_k = s_j^2 \bar{v}_k \).
- The SVD \( (s_j, \bar{v}_j, \bar{u}_j) \) of the matrix \( W \) approximates the singular system \( (\sigma_j, \mu_j, \nu_j) \) of the operator \( L \), where \( \bar{v}_j \) and \( \bar{u}_j \) are discretizations of \( \mu_j \) and \( \nu_j \), respectively.
Approximation of Ensemble Controls

- If \( \hat{\xi} \in \mathbb{R}^{nP} \) is given by \( \hat{\xi}_j = \xi(\beta_j) \) for a function \( \xi \in \mathcal{H}_K \), the minimum norm solution \( \hat{g}^* \) that satisfies \( W\hat{g} = \hat{\xi} \) is given by \( \hat{g}^* = W'z \) where \( WW'z = \hat{\xi} \).
- Basic properties of the SVD yield, for \( mq \leq P \),

\[
\hat{g}^* = \sum_{j=1}^{mq} \frac{\hat{\xi}' \bar{u}_j}{s_j} \bar{v}_j.
\]

- The components of the synthesized minimum norm control \( \hat{u}^* = (\hat{u}_1^*, \ldots, \hat{u}_m^*)' \) are given by

\[
\hat{u}_k^* = \sum_{j=1}^{q} \frac{\hat{\xi}' \bar{u}_{k+m(j-1)}}{s_{k+m(j-1)}} \bar{v}_{k+m(j-1)}.
\]

Computational Issues

- Time and parameter discretizations $N$ and $P$ must be chosen such that $nP < mN$. Then $(W, \hat{\xi})$ represents an underdetermined system.

- To prevent conditioning errors from dominating, choose $q$ so that $s_1/s_{mq} < 10^4$.

- Picard criterion asserts $\exists$ an $\mathcal{H}_T$ solution to $Lu = \xi$ only if $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} |\langle \xi, \nu_n \rangle K|^2 < \infty$. Problematic to check this asymptotic property numerically.
Example: Harmonic Oscillator Ensemble

Consider a system

\[
\frac{d}{dt} \begin{bmatrix} x(t, \omega) \\ y(t, \omega) \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x(t, \omega) \\ y(t, \omega) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
\]

where \( \omega \in K = [\omega_1, \omega_2] \subset \mathbb{R} \), the instantaneous state is \( X(\cdot, \omega) = (x(\cdot, \omega), y(\cdot, \omega))' \in \mathcal{H}_K \), and the control vector is \( U = (u_1, u_2)' \in \mathcal{H}_T \).
Example: Harmonic Oscillator Ensemble

For $X_0(\omega) = (1, 0)'$ and $X_F(\omega) = (0, 0)'$ with $\omega \in [-10, 10]$ and $T = 1$, use $N = 20000$, $P = 20$. 

![Graph showing control synthesis for linear systems example](image-url)
Sensitivity Analysis

The simulation is repeated for different values of time horizon $T$ and time step $\delta$, where $N = T/\delta$, and $P = 40$ is used. (a) Log-log plot of norm of error in the final state as a function of $1/\delta = N/T$. The slope of the lines is close to 1, so that the error is proportional to $\delta$. The lines correspond to $T = 0.1, 0.5, 2, 1, \text{and } 5$, from top to bottom, hence a longer time horizon does not necessarily result in improvement. (b) Significant singular values $q$ as a function of $T$. 
Example: Nonuniform State Transfer

$X_0$ and $X_F$ are star and leaf shaped images, $\omega \in [-10, 10]$, $T = 40$. Use $N = 20000$ and $P = 89$. RMSE in final states is 0.0123, and maximum absolute error is 0.0273.
Example: Nonuniform State Transfer

$X_0$ and $X_F$ are logos of the AFOSR and USAF,

$\omega \in [-10, 10]$, $T = 100$. Use $N = 20000$ and

$P = 277$. RMSE in final states is $5.4068 \times 10^{-4}$, and

maximum absolute error is $8.7394 \times 10^{-4}$. 
Example: Optimal Quantum Transport

A quantum ensemble transport system is given by

\[ dX = A(t, \omega)X + Bu, \]

\[
A(t, \omega) = \begin{bmatrix}
0 & 1 & 0 \\
-\omega^2 & 0 & \omega^2 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]

where \( \omega \in [0.5, 1] \), \( T = 25 \), and \( X_0 = (0, 0, 0)' \) and \( X_F = (1, 0, 1)' \) are initial and target states.

Example: Optimal Quantum Transport

Nominal control, $X_F=(1,0,1)$, $\omega=[0.5,1]$, $T=25$

Ensemble control, $X_F=(1,0,1)$, $\omega=0.75$, $T=25$

Log$_{10}$ of Singular Values

$N = 25000$ and $P = 26$ are used.
Example: Optimal Quantum Transport

Here \( X_F = (1, 0, 0)' \), and \( N = 25000 \) and \( P = 26 \) are used.
Autonomous bilinear ensemble system

Consider the system

\[ \dot{X} = A(\beta)X + \left[ \sum_{i=1}^{m} u_i(t)B_i(\beta)X \right] + B_0(\beta)u(t), \]  \hspace{1cm} (\Gamma)

where \( u = (u_1, \ldots, u_m)' \), \( X = X(t, \beta) \), and \( A(\beta) \in \mathbb{R}^{n \times n} \) and \( B_i(\beta) \in \mathbb{R}^{n \times n}, B_0(\beta) \in \mathbb{R}^{n \times m} \) have elements that are real \( L_\infty \) and \( L_2 \) functions, respectively, defined on a compact set \( K \), and are denoted \( A \in L^{n \times n}_{\infty}(K) \) and \( B_i \in L^{n \times n}_{2}(K), B_0 \in L^{n \times m}_{2}(K) \). This can be rewritten as

\[ \dot{X} = A(\beta)X + \left[ \sum_{j=1}^{n} x_j \tilde{B}_j(\beta) + B_0(\beta) \right] u(t), \]  \hspace{1cm} (\Gamma')
Fixed point control synthesis

Given an estimate \((X^\alpha, u^\alpha)\) of the trajectory-control pair where \(X^\alpha = (x^\alpha_1, \ldots, x^\alpha_n)'\), \(u^\alpha = (u^\alpha_1, \ldots, u^\alpha_m)'\) and \(\alpha\) is an iteration index, substituting \(x_j(t, \beta) = x_j^\alpha(t, \beta)\) for \(j = 1, \ldots, n\) in \((\Gamma')\) yields the time-varying linear system

\[
\dot{X}(t, \beta) = A(\beta)X(t, \beta) + B^\alpha(t, \beta)u(t),
\]

\[
B^\alpha(t, \beta) = \sum_{j=1}^{n} x_j^\alpha(t, \beta) \tilde{B}_j(\beta) + B_0(\beta).
\]

Method for linear systems yields control \(u^{\alpha+1}\) that solves

\[
(L^\alpha u)(\beta) = \int_0^T \Phi(0, \sigma, \beta) B^\alpha(\sigma, \beta)u(\sigma)d\sigma = \xi(\beta),
\]

where \(\xi(\beta) = \Phi(0, T, \beta)X_F(\beta) - X_0(\beta)\). Applying to \(\Gamma'\) with \(X(0, \beta) = X_0(\beta)\) yields \((X^{\alpha+1}, u^{\alpha+1})\).
Fixed point control synthesis

- Repeat until \( E(\alpha) := \|u^{\alpha+1} - u^{\alpha}\|/\|u^{\alpha}\| < \gamma \).
- To avoid divergence, \( \hat{u}^{\alpha+1} \) and \( \hat{X}^{\alpha+1} \) must be obtained using same discretization scheme \( \{\beta_j\} \) and \( \{t_k\} \) for all iterations.
- Use matrix operations to approximate \( \hat{X}^{\alpha+1}(t, \beta) \) by Riemann quadrature.
- This fixed point procedure converges given appropriate regularity conditions.
Example: Bloch System Ensemble

The Bloch System is a 3-dimensional bilinear system:

\[
\begin{bmatrix}
\dot{x}_1(t, \omega, \varepsilon) \\
\dot{x}_2(t, \omega, \varepsilon) \\
\dot{x}_3(t, \omega, \varepsilon)
\end{bmatrix}
= \begin{bmatrix}
0 & -\omega & \varepsilon u \\
\omega & 0 & -\varepsilon v \\
-\varepsilon u & \varepsilon v & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
x_1(t, \omega, \varepsilon) \\
x_2(t, \omega, \varepsilon) \\
x_3(t, \omega, \varepsilon)
\end{bmatrix}
\end{bmatrix}
\]

The Bloch Equations can be re-written as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix}
= \begin{bmatrix}
0 & -\omega & 0 \\
\omega & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \varepsilon \begin{bmatrix}
x_3 & 0 \\
0 & -x_3 \\
-x_1 & x_2
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
\]

This has form \( \dot{X} = A(\omega)X + \varepsilon B(X)U \).
Example: Bloch System Ensemble

Bloch system is ensemble controllable w.r.t. $\omega$ and $\varepsilon$. However, $\dot{X} = A(\omega)X + \varepsilon B^\alpha(t, \omega)u$ is not ens. controllable w.r.t. $\varepsilon$.

Suppose $\omega \in [-\mu, \mu]$, $\varepsilon \equiv 1$, to be steered from $X_0 = (0, 0, 1)'$ to $X_F = (1, 0, 0)'$ in $T = 1$ (A broadband $\pi/2$ pulse).

*Left:* Sampled on $\omega \in [-8, 8]$. Control for $\omega \in [-5, 5]$, $N = 5000$, and $P = 20$ for a $90^\circ$ transfer in $T = 1$ (gray), and nominal system with $\omega = 0$ (black). *Right:* $\omega \in [-1, 1]$, with $N = 2000$ and $P = 20$, looking down.
Example: Bloch System Ensemble

*Top:* Ensemble controls for several values of $\mu$ generated using $N = 5000$, $P = 20$, $T = 1$ for a $90^\circ$ transfer in $T = 1$. *Middle:* Log plot of relative change $E(\alpha)$ between successive control iterates, for $\mu = 0.5$, 1, and 2. *Bottom:* Log-log plot of terminal error as a function of discretization $N$ for $\mu = 0.5$, 1, and 2. Note that the terminal error is smallest for $\mu = 1$. 

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Discussion and Future Work

- Numerical test for Ensemble Controllability
- Ensemble control synthesis for constrained problems
- Necessary and sufficient conditions for convergence of iterative method
- Ensemble control synthesis for general nonlinear systems

Frequency control of oscillating systems

- Spiking of neurons, metabolic chemical reaction systems, circadian pacemakers, semiconductor lasers, vibrating systems, chemical reactors
- Applications include deep brain stimulation, cardiac pacemaking, impedance spectroscopy, and others in physics, chemistry, biology, electrical and mechanical engineering
- *Entrainment* is the *asymptotic* synchronization of an oscillator to an external signal (phase locking)
- Want to design entrainment controls that are optimal, e.g. minimum power, fastest to phase-locking, etc.


Hodgkin-Huxley Equations (1952)

\[
\begin{align*}
c\dot{V} &= I_b + I(t) - \bar{g}_{Na} h(V - V_{Na}) m^3 - \bar{g}_K (V - V_k) n^4 - \bar{g}_L (V - V_L), \\
\dot{m} &= a_m(V) (1 - m) - b_m(V) m, \\
\dot{h} &= a_h(V) (1 - h) - b_h(V) h, \\
\dot{n} &= a_n(V) (1 - n) - b_n(V) n, \\
\end{align*}
\]

\[
\begin{align*}
a_m(V) &= 0.1(V + 40)/(1 - \exp(-(V + 40)/10)), \\
b_m(V) &= 4 \exp(-(V + 65)/18), \\
a_h(V) &= 0.07 \exp(-(V + 65)/20), \\
b_h(V) &= 1/(1 + \exp(-(V + 35)/10)), \\
a_n(V) &= 0.01(V + 55)/(1 - \exp(-(V + 55)/10)), \\
b_n(V) &= 0.125 \exp(-(V + 65)/80). \\
\end{align*}
\]

\[V_{Na} = 50 \text{ mV}, V_K = -77 \text{ mV}, V_L = -54.4 \text{ mV}, \bar{g}_{Na} = 120 \text{ mS/cm}^2, \bar{g}_K = 36 \text{ mS/cm}^2, \bar{g}_L = 0.3 \text{ mS/cm}^2, I_b = 10 \mu\text{A/cm}^2, c = 1 \mu\text{F/cm}^2.\]
We consider a smooth dynamical system $\dot{x} = f(x, u)$, where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is a control.

The system has an attractive, non-constant limit cycle $\gamma(t) = \gamma(t + T)$ satisfying $\dot{\gamma} = f(\gamma, 0)$ on the periodic orbit $\Gamma = \{ y \in \mathbb{R}^n : y = \gamma(t), t \in [0, T] \}$.

Model reduction from a dynamical system on $\mathbb{R}^n$ to a scalar ODE

<table>
<thead>
<tr>
<th>State space model</th>
<th>Phase model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{x} = f(x, u)$</td>
<td>$\dot{\psi} = \omega + Z(\psi)u$</td>
</tr>
<tr>
<td>State: $x \in \mathbb{R}^n$</td>
<td>phase: $\psi \in [0, 2\pi)$</td>
</tr>
<tr>
<td>control: $u \in \mathbb{R}$</td>
<td></td>
</tr>
</tbody>
</table>

$\omega \equiv$ natural frequency and $Z \equiv$ phase response curve (PRC)
Phase Models

- $\dot{\psi} = \omega + Z(\psi)u$ is a valid representation of $\dot{x} = f(x, u)$ for $u$ such that $x$ remains within a neighborhood of $\Gamma$.

- The *phase response curve* (PRC) $Z$ is the infinitesimal sensitivity of the phase to an external input at a given phase.

- All points $x$ on an *isochron* correspond to an *asymptotic phase*.

- The natural frequency $\omega$ and PRC $Z$ are obtained using Floquet theory.

- The PRC can be computed numerically when the system dynamics are known, or can be approximated experimentally by measuring system response to brief, strong pulses.
Frequency control objectives

- Minimum power frequency shift: entrain the system \( \dot{\psi} = \omega + Z(\psi)u \) to a target frequency \( \Omega \) with periodic control \( u(t) = v(\Omega t) \) where \( v \) is \( 2\pi \)-periodic.

- Minimum energy ensemble entrainment: find \( u(t) = v(\Omega t) \) that entrains a collection \( \{ \dot{\psi} = \omega + Z(\psi)u : \omega \in (\omega_{\text{min}}, \omega_{\text{max}}) \} \) of systems with common PRC \( Z \) and natural frequencies on \( (\omega_1, \omega_2) \) to target frequency \( \Omega \).

- Maximum range of entrainment: find control \( u(t) = k(\Omega t) \) that entrains the largest collection of oscillators of form \( \{ \dot{\psi} = \omega + Z(\psi)u : \omega \in (\omega_{\text{min}}, \omega_{\text{max}}) \} \) to target frequency \( \Omega \).

- Fast entrainment: find control \( u(t) = k(\Omega t) \) that achieves phase locking of \( \dot{\psi} = \omega + Z(\psi)u \) to a target frequency \( \Omega \) in the shortest time.

- \( n : m \) entrainment: \( m \) cycles of the oscillator occur for every \( n \) control cycles.
Arnold Tongues

- The region in the power-frequency plane where entrainment of an oscillator by a given waveform occurs is called an Arnold Tongue.
- RMS Power vs. target frequency $\Omega$. Theoretical boundaries (lines) and boundary values obtained by numerical experiments (points)
Control performance

- Arnold tongues for Hodgkin-Huxley model and various controls
- Optimal controls for frequency increase (blue) and decrease (green), sine wave (red), and maximum range (purple). Lines are theoretical, points are computed.
- For a fixed power, blue is rightmost, purple is broadest.
Thanks for your attention!