A numerical condition for the physical realizability of a quantum linear system

Shanon L. Vuglar and Ian R. Petersen

School of Engineering and Information Technology
University of New South Wales at the Australian Defence Force Academy
Canberra, Australia

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Motivation

- Quantum Optics
- Quantum Communications
- Quantum Computing
- Precision Measurement
Coherent Quantum Feedback Control

- Quantum Plant
- Quantum Controller
- Avoids measurement and associated destruction of quantum information
- Physical Realizability / Is implementation possible?
Implementation of a Transfer Function

- Controller Transfer Function normally determines closed loop system performance
- Is a particular transfer function physically realizable as a quantum system?
- A strictly proper transfer function cannot be implemented as a quantum system without additional noise
- Is a particular transfer function physically realizable as a quantum system with the minimum number of additional noises?
- Numerical solution
Implementing Quantum Systems

The following papers give details and algorithms for experimentally implementing physically realizable quantum systems:

- Petersen, “Cascade cavity realization for a class of complex transfer functions arising in coherent quantum feedback control,” *Automatica*, 2011
Introduction

How additional noises are introduced

Cavity

Input

Ignored Output

Unwanted Input (Vacuum Noise)

Output
System Model: LTI Non-commutative Stochastic Quantum system

\[ dx(t) = Ax(t) \, dt + [B_1 \ B] \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}; \quad x(0) = x_0 \]

\[ dy(t) = Cx(t) \, dt + [D_1 \ 0_{ny \times nu}] \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix} \]

- \( A, B_1, B, C, D_1 \) real matrices
- \( x(t) \): vector of non-commutative self adjoint system variables
- \( v(t) \): noise vector (non-commutative Wiener processes)
- \( du(t) = \beta_u(t) \, dt + \tilde{u}(t) \): signal vector, where
  - \( \tilde{u}(t) \): noise
  - \( \beta_u(t) \, dt \): adapted, self-adjoint
Physical Realizability

- System dynamics correspond to collection of open quantum harmonic oscillators.

**Definition**

By the *canonical commutation relations* we mean that the system variables $x$ satisfy the commutation relations

$$[x_i(t), x_j(t)] = 2i \Theta_{ij}$$

where $\Theta$ is a block diagonal matrix with each diagonal block equal to

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  

(James, Nurdin, Petersen, *IEEE Transactions on Automatic Control*, 2008.)
Physical Realizability

Definition
A system is an open quantum harmonic oscillator if:

- $\Theta$ is canonical,
- there exists a Hamiltonian $H = \frac{1}{2}x(0)^T R x(0)$; $R$ real, symmetric, and
- there exists a coupling operator $L = \Lambda x(0)$; $\Lambda$ complex-valued, such that:

$$
A = 2\Theta \left( R + \text{Im} \left( \Lambda^\dagger \Lambda \right) \right);
$$

$$
\begin{bmatrix}
B_1 & B
\end{bmatrix} = 2i\Theta \left[ -\Lambda^\dagger & \Lambda^T \right] \Gamma;
$$

$$
C = P^T \left[ \sum_{n_y} 0 \ \sum_{n_y} \right] \left[ \begin{array}{cc}
\Lambda + \Lambda^\# & \Lambda + \Lambda^\#
\end{array} \right];
$$

$$
\begin{bmatrix}
D_1 & 0_{n_y \times n_u}
\end{bmatrix} = \left[ I_{n_y \times n_y} \ 0_{n_y \times (n_v+n_u-n_y)} \right].
$$

Here: $\Gamma_{n_v \times n_v} = P \text{diag}(M)$; $M = \frac{1}{2} \left[ \begin{array}{cc}
1 \ i \\
1 & -i
\end{array} \right]$; $\sum_{n_y} = \left[ I_{\frac{1}{2} n_y \times \frac{1}{2} n_y} \ 0_{\frac{1}{2} n_y \times \frac{1}{2} (n_v+n_u-n_y)} \right]$;

$P \left[ a_1 \ a_2 \ \cdots \ a_{2m} \right] = \left[ a_1 \ a_3 \ \cdots \ a_{2m-1}a_2 \ a_4 \ \cdots \ a_{2m} \right]$.

(James, Nurdin, Petersen, *IEEE Transactions on Automatic Control*, 2008.)
Previous Results

Consider the system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) dt + \begin{bmatrix} B_1 & B \end{bmatrix} \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}; \quad x(0) = x_0 \\
\frac{dy(t)}{dt} &= Cx(t) dt + \begin{bmatrix} D_1 & 0_{ny \times nu} \end{bmatrix} \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}
\end{align*}
\]

Lemma

and suppose \( A, B \text{ and } C \) are such that \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times nu}, C \in \mathbb{R}^{ny \times n}, \Theta = \text{diag}(J) \) is canonical and the noise variables are also canonical. Then there exists an even integer \( n_v \geq n_y \) and matrix \( B_1 \in \mathbb{R}^{n \times n_v} \) such that the system is physically realizable.

(James, Nurdin, Petersen, IEEE Transactions on Automatic Control, 2008.)
Previous Results

Consider the system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) dt + \begin{bmatrix} B_1 & B \end{bmatrix} \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}; \\
\quad x(0) = x_0 \\
\frac{dy(t)}{dt} &= Cx(t) dt + \begin{bmatrix} D_1 & 0_{ny \times nu} \end{bmatrix} \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}
\end{align*}
\]

Theorem

Consider an LTI system where $A$, $B$ and $C$ are given and the system commutation matrix $\Theta$ is canonical. There exists $B_1$ and $D_1$ such that the system is physically realizable with the number of quantum noises in $dv$ equal to $n_u + 2(n - n_\lambda)$ where $n_\lambda$ is the multiplicity of the least (i.e. most negative) eigenvalue of the matrix $i (\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$. (Vuglar, Petersen, *Australian Control Conference*, 2011)
Previous Results

Consider the system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) \, dt + [B_1 \ B] \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}; \quad x(0) = x_0 \\
\dot{y}(t) &= Cx(t) \, dt + [D_1 \ 0_{ny \times nu}] \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}
\end{align*}
\]

Theorem

Consider an LTI system where \(A, B\) and \(C\) are given and the system commutation matrix \(\Theta\) is canonical. Suppose the Riccati equation

\[
\Psi B \Theta B^T \Psi - A^T \Psi - \Psi A - C^T \Theta C = 0
\]

has a non-singular solution \(\Psi\) which is skew symmetric and suppose that there exists a real non-singular matrix \(T\) such that \(\Psi = T^T \Theta T\). Then there exists a system described by \(\{\tilde{A}, \tilde{B}, \tilde{C}\}\) with the same transfer function as the system \(\{A, B, C\}\) which can be physically realized with the minimum number of additional noises \(n_v = n_u\), where \(\tilde{A} = TAT^{-1}; \tilde{B} = TB; \tilde{C} = CT^{-1}\).

(Vuglar, Petersen, *Australian Control Conference*, 2011)
Main Result

- Conditions for when the Riccati equation has a non-singular skew symmetric solution $\Psi$, of the form $\Psi = T^T \Theta T$ ($T$ non-singular)
- Proof leads to numerical procedure for solving ARE / testing for physical realizability with minimal additional noises.
Main Result

Let $R = -B\Theta B^T$, $Q = C^T\Theta C$, and $X = \Psi$. Then:

$$A^T X + XA + XRX + Q = 0.$$  

Define $H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}$, and $P = -i \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Note: $P^{-1} = P^\dagger$, $(PH)$ is skew symmetric, and $P^{-1}HP = P^\dagger HP = -H^\dagger$. That is, $H$ and $-H^\dagger$ are similar, $\lambda$ is an eigenvalue of $H$ if and only if $-\lambda$ is, i.e. the eigenvalues of $H$ are symmetric about the imaginary axis.
Ric operator

- Assume $H$ has no eigenvalues on the imaginary axis.
- Let $\chi_-(H)$ be the $n$-dimensional spectral subspace of $H$ corresponding to its negative eigenvalues.
- $\chi_-(H) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}; X_1, X_2 \in \mathbb{C}^{n\times n}$.
- Assume $X_1$ is non-singular (equivalently that $\chi_-(H)$ and $\text{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$ are complementary)
- Define $X = X_2X_1^{-1}$.
- $X$ is uniquely determined by $H$; we shall denote this function $Ric$, i.e. $X = Ric(H)$.
- $Ric$ has domain: $\text{dom}(Ric)$ consisting of Hamiltonian matrices $H$ satisfying two properties:
  - that $H$ has no purely imaginary eigenvalues,
  - that $X_1$ is non-singular.
Main Result

Theorem

Suppose $H \in \text{dom}(\text{Ric})$ (that is, $H$ has no purely imaginary eigenvalues, and $X_1$ is non-singular) and $X = \text{Ric}(H)$. Then $X$ is skew-symmetric and solves the algebraic Riccati equation $A^T X + XA + XRX + Q = 0$. 
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Main Result

Corollary

Consider an LTI system with $A$, $B$ and $C$ given and with the canonical commutation matrix $\Theta$ and suppose $H \in \text{dom}(\text{Ric})$ and $X = \text{Ric}(H)$ is non-singular. There exists a system $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ with the same transfer function as the system $\{A, B, C\}$ which is physically realizable with the minimum additional noises $n_v = n_u$, where $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, and $\tilde{C} = CT^{-1}$. 
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Example

We consider an example of classical-quantum controller synthesis from (James, Nurdin, Petersen, *IEEE Transactions on Automatic Control*, 2008).

Particular state space representation given cannot be implemented as a fully quantum system with the minimum number of additional noises.

There exists another state space realization of the same transfer function which can.

Demonstrate the numerical method for solving the Riccati equation, and obtaining the system to be implemented.
Example: Implementing a particular state space representation

Consider the system with

\[ A = \begin{bmatrix} -1.3894 I_{2 \times 2} & -0.4472 I_{2 \times 2} \\ -0.2 I_{2 \times 2} & -0.25 I_{2 \times 2} \end{bmatrix}; \]

\[ B = \begin{bmatrix} -0.4472 I_{2 \times 2} \\ 0_{2 \times 2} \end{bmatrix}; \]

\[ C = \begin{bmatrix} -0.4472 I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}; \]

Here \( n = 4, n_u = 2 \) and \( n_y = 2 \).

The system transfer function is:

\[ G(S) = \begin{bmatrix} \frac{0.2s+0.05}{s^2+1.639s+0.2579} & 0 \\ 0 & \frac{0.2s+0.05}{s^2+1.639s+0.2579} \end{bmatrix}. \]
Example: Implementing a particular state space representation

Applying the *Theorem* (Vuglar, Petersen, *Australian Control Conference*, 2011):

\[
S = i \left( \Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C \right)
\]

\[
= \begin{bmatrix}
0 & 2.3788i & 0 & 0.6472i \\
-2.3788i & 0 & -0.6472i & 0 \\
0 & 0.6472i & 0 & 0.5i \\
-0.6472i & 0 & -0.5i & 0
\end{bmatrix}
\]

- least eigenvalue: \(-2.5802\); multiplicity 1.
- physical realizability requires \(n_v = n_u + 2(n - n_\lambda) = 8\) quantum noises
- strictly greater than the minimum number of additional quantum noises \(n_v = n_u = 2\).
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Example

Example: Implementing the transfer function

\[ R = \begin{bmatrix} 0 & -0.2 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad Q = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \]

\[ H = \begin{bmatrix} -1.389 & 0 & -0.447 & 0 & 0 & -0.2 & 0 & 0 \\ 0 & -1.389 & 0 & -0.447 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 1.389 & 0 & 0.2 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 1.389 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0.447 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.447 & 0 & 0.25 \end{bmatrix}. \]

Eigenvalues of H: \(-1.4496, -1.4496, -0.1745, -0.1745, 0.1745, 0.1745, 1.4496, 1.4496\).
Example: Implementing the transfer function

Form $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ by stacking the eigenvectors corresponding to the eigenvalues of $H$ with negative real parts:

$$
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix}
-0.3520 & -0.9838 & 0.0223 & -0.0409 \\
0 & 0 & 0.9835 & 0.3496 \\
0.93290 & -0.1640 & 0.0037 & 0.1083 \\
0 & 0 & 0.1640 & -0.9266 \\
0.0520 & 0.0706 & -0.0016 & 0.0060 \\
0 & 0 & -0.0186 & -0.0544 \\
-0.0548 & -0.0186 & 0.0004 & -0.0064
\end{bmatrix}
$$
Example: Implementing a transfer function

$X_1$ is non-singular, so we can apply our *Theorem*:

$$X = X_2X_1^{-1} = \begin{bmatrix}
0 & 0.0763 & 0 & -0.0270 \\
-0.0763 & 0 & 0.0270 & 0 \\
0 & -0.0270 & 0 & 0.0486 \\
0.0270 & 0 & -0.0486 & 0
\end{bmatrix}$$

is a skew-symmetric solution to the Riccati equation, as can be confirmed by substitution.
Example: Implementing a transfer function

Finally, since $X$ is non-singular we can apply our Corollary: There exists a system $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ with the same transfer function $G(s)$ as the system $\{A, B, C\}$ which is physically realizable with the minimum additional noises $n_v = n_u = 2$ where

$$X = T^T \Theta T; \quad T = \begin{bmatrix} 0 & 0.2599 & 0 & -0.1587 \\ -0.2599 & 0 & 0.1587 & 0 \\ -0.0933 & 0 & -0.1529 & 0 \\ 0 & -0.9033 & 0 & -0.1529 \end{bmatrix};$$

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -0.7922 & 0 & 0 & 1.3232 \\ 0 & -0.7922 & -1.3232 & 0 \\ 0 & -0.3132 & -0.8472 & 0 \\ 0.3123 & 0 & 0 & -0.8472 \end{bmatrix};$$

$$\tilde{B} = TB = \begin{bmatrix} 0 & -0.1162 \\ 0.1162 & 0 \\ 0.0417 & 0 \\ 0 & 0.0417 \end{bmatrix};$$

$$\tilde{C} = CT^{-1} = \begin{bmatrix} 0 & 1.2533 & 1.3010 & 0 \\ -1.2533 & 0 & 0 & 1.3010 \end{bmatrix}.$$

Remark: It is always possible to factorise $X$ as shown and our paper provides a method for finding $T$. 
Conclusion

- An arbitrary LTI system can always be implemented as a quantum system if we allow additional quantum noises.
- Incorporating unnecessary noises is undesirable.
- Previously we obtained an expression for an upper bound on the number of required additional noises.
- Our main result here is to provide a numerical method for determining if a given transfer function matrix can be implemented as a linear quantum system with minimal additional noises.
- Direct applications include coherent quantum feedback control where we wish to know whether it is possible to implement a given controller using the minimum number of additional quantum noises.
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Questions?