The Descriptor Discrete–Time Riccati Equation: Numerical Solution and Applications

Cristian Oară
Dep. of Automatic Control and Systems Engineering, “Politehnica” Univ. Bucharest
cristian.oara@acse.pub.ro

Raluca Andrei
Delft Center for Systems and Control
Delft Univ. of Technology
r.m.andrei@tudelft.nl

*This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS project number PN-II-ID-PCE-2011-3-0235.
Outline

1. The Descriptor Discrete–Time Algebraic Riccati Equation (DDTARE)

2. Descriptor Symplectic Pencils and Deflating Subspaces

3. Computation of Solutions to DDTARE

4. Applications to General Systems (Nonproper/Polynomial/Any Poles, Zeroes, Rank):
   - Rank Compression (Squaring Down) with $L^\infty$–Norm Preservation
   - $(J, J')$–Spectral Factorization

5. Conclusions
1. The Descriptor Discrete–Time Algebraic Riccati Equation (1/2)

**DDTARE:**

\[
E^*XE - A^*XA - ((E - A)^*XB + L)R^{-1}(L^* + B^*X(E - A)) + Q = 0
\]

where \( E \in \mathbb{C}^{n \times n} \), \( A \in \mathbb{C}^{n \times n} \), \( B \in \mathbb{C}^{m \times m} \), \( L \in \mathbb{C}^{n \times m} \), \( Q = Q^* \in \mathbb{C}^{n \times n} \), \( R = R^* \in \mathbb{C}^{m \times m} \), \((R \text{ invertible})\).

**Remarks:**

- The DDTARE is not a standard Riccati equation (sort of a mixture between a CTARE and a DTARE)
- We do not make any positivity assumptions on the matrix coefficients
- The origin: A quadratic optimization problem for a dynamical system given by a realization centered in 1
1. The Descriptor Discrete–Time Algebraic Riccati Equation (2/2)

The DDTARE has many solutions but we are only interested in two types:

**Definition 1.** \( X = X^* \in \mathbb{C}^{n \times n} \) is called a **stabilizing / antistabilizing** solution to the DDTARE if

\[
\Lambda(z(E + BF) - (A + BF)) \subset D / \quad \Lambda(z(E + BF) - (A + BF)) \subset \mathcal{C} - \mathcal{D},
\]

where

\[
F := -R^{-1}(L^* + B^*X(E - A))
\]

is called the **stabilizing / antistabilizing** Riccati feedback.

The Theorem below states the uniqueness of the stabilizing (antistabilizing) solution, if one exists.

**Theorem 2.** *If the DDTARE has a stabilizing / antistabilizing solution then it is unique.*
2. Descriptor Symplectic Pencils and Deflating Subspaces (1/4)

We introduce two matrix pencils associated with the DDTARE to be used for characterization and computation of the solutions.

The $(2n + m) \times (2n + m)$ Descriptor Symplectic Pencil (DSP):

$$z \left[ \begin{array}{ccc} E & 0 & B \\ Q & A^* & L \\ 0 & 0 & 0 \end{array} \right] - \left[ \begin{array}{ccc} A & 0 & B \\ Q & E^* & L \\ L^* & B^* & R \end{array} \right]$$

The $2n \times 2n$ Reduced Descriptor Symplectic Pencil (RDSP):

$$z \left[ \begin{array}{cc} E - BR^{-1}L^* & -BR^{-1}B^* \\ Q - LR^{-1}L^* & A^* - LR^{-1}B^* \end{array} \right] - \left[ \begin{array}{cc} A - BR^{-1}L^* & -BR^{-1}B^* \\ Q - LR^{-1}L^* & E^* - LR^{-1}B^* \end{array} \right]$$
2. Descriptor Symplectic Pencils and Deflating Subspaces (2/4)

Let \( zM - N \) be a regular matrix pencil (square and \( \det(zM - N) \neq 0 \)). Deflating subspaces are the natural extension of invariant subspaces of a matrix to the pencil case:

The linear space \( V \subset \mathbb{C}^n \) is a deflating subspace if

\[
\dim(MV + NV) = \dim(V).
\]

The following result characterizes deflating subspaces in terms of basis matrices:

**Theorem 3.** Let \( zM - N \) be a regular \( n \times n \) matrix pencil.

1. If \( V = \text{Im}(V) \) is an \( \ell \)–dimensional deflating subspace with \( V \) basis matrix, then there is a regular \( \ell \times \ell \) pencil \( zT - S \) equivalent to \( (zM - N)|_V \), s. t.

\[
MVS = NVT. \tag{1}
\]

2. Conversely, if (1) holds for a certain \( n \times \ell \) basis matrix \( V \) and a regular \( \ell \times \ell \) pencil \( zT - S \), then \( V = \text{Im}(V) \) is a deflating subspace of \( zM - N \) and \( zT - S \) is equivalent to \( (zM - N)|_V \).

\( (zM - N)|_V \) is the map restricted to \( V \)
2. Descriptor Symplectic Pencils and Deflating Subspaces (3/4)

Assume $E - A$ and $R$ are invertible. Then\(^2\):

1. The DSP and RDSP are regular pencils.

2. The DSP has $n^\infty = n_0 + m$ and $n^{<1} = n^{>1} + n^0$.

3. The RDSP has $n^\infty = n^0$ and $n^{<1} = n^{>1} + n^\infty$.

4. If $n^1 = 0$ then the DSP has $n^{<1} = n$ and $n^{>1} + n^\infty = n + m$ while the RDSP has $n^{<1} = n$ and $n^{>1} + n^\infty = n$.

- A DSP has an $n$ dimensional deflating subspace with stable spectrum (included in $D$) if and only if $n^1 = 0$

- The RDSP has an $n$ dimensional deflating subspace with stable spectrum if and only if it has an $n$ dimensional deflating subspace with antistable (in $C - D$) spectrum if and only if $n^1 = 0$.

\(^2\)\(n^{>1}, n^{<1}, n^1, n^0, n^\infty\) are the number of generalized eigenvalues with modulus strictly greater than one, strictly less than one, equal to 1 equal to 0, and at infinity (everywhere multiplicities counting).
2. Descriptor Symplectic Pencils and Deflating Subspaces (4/4)

We have the following useful characterization of stable / antistable deflating subspaces:

- Assume the DSP has an $n$–dimensional stable deflating subspace with basis matrix

\[ V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \]

Then

\[ V_1^*(E - A)^*V_2 = V_2^*(E - A)V_1. \]  \hspace{1cm} (3)

- If the RDSP has an $n$–dimensional stable (or antistable) deflating subspace with basis matrix

\[ V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]

then (3) holds true as well.

Interestingly, (3) does not necessary hold true for a basis matrix (2) of an $n$–dimensional antistable deflating subspace of the DSP.
3. Computation of Solutions to DDTARE (1/2)

The next two theorems give support for computing the stabilizing / antistabilizing solutions to the DDTARE.

**Theorem 4.** Assume $E - A$ is invertible. The following two assertions are equivalent:

1. $R$ is invertible and the DDTARE has a stabilizing solution;

2. The DSP is regular and has an $n$–dimensional stable deflating subspace with basis matrix

   $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}_{n \times m}$

   having $V_1$ invertible.

   The stabilizing solution can be computed from

   $$X = V_2 V_1^{-1} (E - A)^{-1}$$

   and the stabilizing Riccati feedback can be computed from

   $$F = V_3 V_1^{-1}.$$
3. Computation of Solutions to DDTARE (2/2)

**Theorem 5.** Assume \( E - A \) is invertible. The following two assertions are equivalent:

1. \( R \) is invertible and the DDTARE has a stabilizing / antistabilizing solution;

2. The RDSP is regular and has an \( n \)-dimensional stable / antistable deflating subspace with basis matrix

\[
V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_{n}
\]

having \( V_1 \) invertible.

The stabilizing / antistabilizing solution can be computed from

\[
X = V_2 V_1^{-1} (E - A)^{-1}.
\]

**Remarks:**
- The computational burden lies in the computation of a maximal stable / antistable deflating subspace for which efficient numerical algorithms based on unitary transformations are available.
- It is advisable to use the DSP instead of the RDSP whenever possible as its coefficients do not contain possibly ill–conditioned matrix inversion. Unfortunately, the antistabilizing solution can be computed solely from the RDSP.
4. Applications (1/5)

We show the role of the DDTARE in the solution of two control problems for a completely general system given by an irreducible generalized state–space representation

\[
G(z) = C(zE - A)^{-1}B + D =: \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix}.
\]

(A) Squaring down with \( L^\infty \) norm preservation. Squaring–down the \( p \times m \) system (6) with \( L^\infty \) norm preservation means to design a postcompensator \( G_{post}(z) \) and a precompensator \( G_{pre}(z) \) such that

\[
\tilde{G}_{sqd}(z) := G_{post}(z)G(z)G_{pre}(z) = \begin{bmatrix} G_{sqd}(z) & 0 \\ 0 & 0 \end{bmatrix},
\]

where \( G_{sqd}(z) \) is square, invertible, with the same \( L^\infty \) norm as the original system \( G(z) \).

Remarks: • We consider the general case in which \( p, m, \text{rank}(G(z)) \) are all arbitrary

• We give a solution only to the postcompensation problem (full row rank compression) since the solution for the precompensation follows by duality.

• The problem may be seen alternatively as a rank revealing compression of a rational matrix by an isometric matrix.
4. Applications (2/5)

(B) The general \((J, J')\)–spectral factorization problem. This problem for (6) consists in finding a system \(\Pi(z)\) which has full row normal rank and only marginally stable zeros \(s.\ t.\)

\[
G^\#(z)JG(z) = \Pi^\#(z)J'\Pi(z),
\]

where \(G(z)\Pi^{(+)}(z)\) has no poles on the unit circle. Here \(G^\#(z) := G^\ast\left(\frac{1}{z}\right)\), \(\Pi^{(+)}(z)\) stands for the Moore–Penrose pseudoinverse of \(\Pi(z)\), and \(J\) and \(J'\) are two constant signature matrices.

Remarks: • The novelty is that we allow a general discrete–time system (6) having any type of singularity, including arbitrary normal rank, poles and zeroes at infinity, at zero, or on the unit circle.

• The result may be applied in particular to spectral factorization of a general polynomial system.
4. Applications (3/5)

The solutions to both problems use a special decomposition of the irreducible generalized state–space realization of the system to start with, but with specific properties for each case. Precisely, there are three orthogonal matrices $U$, $Q$ and $Z$ s. t.

$$
\begin{bmatrix}
U & 0 \\
0 & Q
\end{bmatrix}
\begin{bmatrix}
A - zE \\
C
\end{bmatrix}
\begin{bmatrix}
B \\
D
\end{bmatrix}
= Z
$$

where the matrices $A_{\ell} - E_{\ell}$ and $B_n$ are invertible, $D_{\ell}$ has full column rank, the pair $[A_{\ell} - zE_{\ell} \ B_{\ell}]$ is controllable and either of the following two properties hold:

(A) The pencil $A_{rz} - zE_{rz}$ has full row normal rank and the pencil

$$
\begin{bmatrix}
A_{rz} - zE_{rz} & * & * & * \\
0 & A_{\ell} - zE_{\ell} B_{\ell}(1 - z) & * & * \\
0 & 0 & 0 & B_n \\
0 & C_{\ell} & D_{\ell} & *
\end{bmatrix}
$$

has full column rank $\forall z \in C$.

(B) The pencil $A_{rz} - zE_{rz}$ has full row rank in $D_c$, $E_{rz}$ has full row rank, and the pencil (9) has full column rank in $D$. 
4. Applications (4/5)

**Theorem 6. [L∞—norm preserving rank compression ]** Let (6) be an irreducible
generalized state–space system and $U$, $Q$ and $Z$ the three orthogonal matrices for
which (8) holds with property (A). The DDTARE

$$
E_{\ell}^* X E_{\ell} - A_{\ell}^* X A_{\ell} - ((E_{\ell} - A_{\ell})^* X B_{\ell} + C_{\ell}^* D_{\ell}) \\
\times (D_{\ell}^* D_{\ell})^{-1} (D_{\ell}^* C_{\ell} + B_{\ell}^* X (E_{\ell} - A_{\ell})) + C_{\ell}^T C_{\ell} = 0 
$$

(10)

has an invertible stabilizing symmetric solution $X_s$. An $L^\infty$ norm–preserving full row
rank compression and the squared–down system are given by

$$
G_{\text{post}}(z) = I - (C_{\ell} + D_{\ell} F_s)(z E_{\text{post}} - A_{\text{post}})^{-1} B_{\text{post}}(1 - z), 
$$

(11)

$$
G_{\text{sqd}}(z) = D_{\text{sqd}} + C_{\text{sqd}} (z E - A)^{-1} B 
$$

(12)

where

$$
B_{\text{post}} := -X_s^{-1} (E_{\ell} - A_{\ell})^{-*} (C_{\ell} + D_{\ell} F_s), \\
z E_{\text{post}} - A_{\text{post}} := z (E_{\ell} - B_{\text{post}} C_{\ell}) - (A_{\ell} - B_{\text{post}} C_{\ell}), \\
\begin{bmatrix}
C_{\text{sqd}} & D_{\text{sqd}}
\end{bmatrix} := \begin{bmatrix}
0 & -D_{\ell} F_s & D_{\ell} & 0
\end{bmatrix} Z^*,
$$

and $F_s$ is the Riccati stabilizing feedback.
Theorem 7. [(J, J')–spectral factorization] Let \( G(z) \) be a general rmf given by an irreducible realization (6), and let \( U, Q \) and \( Z \) be three constant unitary matrices such that (8) holds with property (B). The \((J, J')\)–spectral factorization problem has a solution if and only if there is a constant invertible matrix \( V \) such that

\[
D^* \ell JD_\ell = V^* J'V
\]  

(13)

and the DDTARE

\[
E^* \ell XE_\ell - A^* \ell XA_\ell - ((E_\ell - A_\ell)^* XB_\ell + C^* \ell JD_\ell) \\
\times (D^* \ell JD_\ell)^{-1}(B^* \ell X(E_\ell - A_\ell) + D^* \ell JC_\ell) + C^* \ell JC_\ell = 0
\]

(14)

has an invertible stabilizing solution \( X_s \). The \((J, J')\) spectral factor is given by

\[
\Pi(z) := D_{\text{spec}} + C_{\text{spec}}(zE - A)^{-1}B,
\]  

(15)

where \([ \begin{array}{cc} C_{\text{spec}} & D_{\text{spec}} \end{array} ] := [ \begin{array}{cc} 0 & -VF_s \\ V & 0 \end{array} ] \) \( Z^* \) and \( F_s \) is the Riccati stabilizing feedback.
5. Conclusions (1/1)

- The DDTARE plays a role also in the solutions to the \((J, J')\)-lossless factorization of a completely general system and the zero compensation problem by cascade connection.

- The computation of the stabilizing solution to the DDTARE is based on stable unitary transformations and avoids matrix inversion until the final stage of the computation.

- One can extend the DDTARE to a DDTARS(ystem) which allows a full rank decomposition of a general rational matrix function.